

———— X-Sun-Data-Type: default X-Sun-Data-Description: default X-Sun-Data-Name: qgroups.
Sun-Charset: us-ascii X-Sun-Content-Lines: 527

UCLA/99/TEP/47
February 2000

QUANTUM GROUPS AND FIELD THEORY

R. J. Finkelstein

*Department of Physics and Astronomy
University of California, Los Angeles, CA 90095-1547*

Abstract. When the symmetry of a physical theory describing a finite system is deformed by replacing its Lie group by the corresponding quantum group, the operators and state function will lie in a new algebra describing new degrees of freedom. If the symmetry of a field theory is deformed in this way, the enlarged state space will again describe additional degrees of freedom, and the energy levels will acquire fine structure. The massive particles will have a stringlike spectrum lifting the degeneracy of the point-particle theory, and the resulting theory will have a non-local description. Theories of this kind naturally contain two sectors with one sector lying close to the standard theory while the second sector describes particles that should be more difficult to observe.

PACS numbers: 81R50, 81T13

1. Introduction.

Since the Lie groups may be considered as degenerate forms of the quantum groups,¹ it may be of interest to generalize the symmetry of a physical theory by replacing its Lie group by the corresponding quantum group.² When this generalization is attempted for the quantum mechanics of finite systems, such as the harmonic oscillator or the hydrogen atom, it is found that the state space must be expanded to describe additional degrees of freedom in a manner rather similar to the way that spin fine structure is added to an atomic structure.³ When the corresponding generalization of field theory is carried out, the expanded state space may be interpreted to describe additional degrees of freedom associated with the spatial extension of the field quanta, or elementary particles.³ While the fine structure of stringlike theories results from explicit geometrical postulates about non-local structure, the generalization of gauge theories to quantum groups leads to fine structure without the necessity of supplementary geometric input. If successful this procedure would resemble the manner in which the algebraic formulation of Pauli spin replaces the geometric picture of a spinning electron.

2. The Quantum Group $SL_q(2)$.

We base our work on the simplest non-trivial example: $SL_q(2)$. Its 2-dimensional representation, T , may be defined as follows:

$$T\epsilon T^t = T^t\epsilon T = \epsilon \quad (2.1)$$

$$\det_q T = 1 \quad (2.2)$$

where ϵ is a 2-dimensional representation of the imaginary unit:

$$\epsilon = \begin{pmatrix} 0 & q_1^{1/2} \\ -q_1^{1/2} & 0 \end{pmatrix} \quad q_1 = q^{-1} . \quad (2.3)$$

Here T^t is the transposed matrix and $\det_q T$ is defined by

$$\epsilon_{ij} T_{im} T_{jn} = \epsilon_{mn} \det_q T . \quad (2.4)$$

Set

$$T = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} . \quad (2.5)$$

Then by (2.1) and (2.2)

$$\begin{aligned} \alpha\beta &= q\beta\alpha & \alpha\gamma &= q\gamma\alpha & \alpha\delta - q\beta\gamma &= 1 & \beta\gamma &= \gamma\beta \\ \delta\beta &= q_1\beta\delta & \delta\gamma &= q_1\gamma\delta & \delta\alpha - q_1\beta\gamma &= 1 \end{aligned} \quad (2.6)$$

Consider now a matrix realization of the algebra (2.6) and set

$$\delta = \bar{\alpha} , \quad \beta = \bar{\beta} \quad \text{and} \quad \gamma = \bar{\gamma} \quad (2.7)$$

where the bar signifies Hermitian conjugation.

Then

$$\begin{aligned}\alpha\beta &= q\beta\alpha & \alpha\gamma &= q\gamma\alpha & \alpha\bar{\alpha} - q\beta\gamma &= 1 & \beta\gamma &= \gamma\beta \\ \bar{\alpha}\beta &= q_1\beta\bar{\alpha} & \bar{\alpha}\gamma &= q_1\gamma\bar{\alpha} & \bar{\alpha}\alpha - q_1\beta\gamma &= 1\end{aligned}\quad (2.8)$$

Under Hermitian conjugation, Eqs. (2.8) imply

$$q = \bar{q} \quad (2.9)$$

so that q is a real number. Since β and γ commute, they have a common set of eigenstates $|nm\rangle$. Let the ground states of β and γ be $|om\rangle$ and $|no\rangle$ respectively, where

$$\begin{aligned}\alpha|om\rangle &= \alpha|no\rangle = 0 \\ \beta|om\rangle &= b|om\rangle \\ \gamma|no\rangle &= c|no\rangle.\end{aligned}\quad (2.10)$$

Define the state $|nm\rangle$ by the recursive relations:

$$\bar{\alpha}|nm\rangle = \lambda_{nm}|n+1, m+1\rangle \quad (2.11)$$

$$\alpha|nm\rangle = \mu_{nm}|n-1, m-1\rangle. \quad (2.12)$$

By (2.8) and (2.10)

$$\beta|nm\rangle = q^n b|nm\rangle \quad (2.13)$$

$$\gamma|nm\rangle = q^m c|nm\rangle \quad (2.14)$$

$$bc = -q \quad (2.15)$$

$$|\lambda_{nm}|^2 = 1 - q^{n+m+2} \quad (2.16)$$

$$|\mu_{nm}|^2 = 1 - q^{n+m} \quad (2.17)$$

By the preceding equations

$$q < 1. \quad (2.18)$$

3. Mass Spectrum in a Global Theory.

Suppose that there is a mass term of the following form:

$$M\tilde{\psi}\epsilon\psi \quad (3.1)$$

where ψ is a fundamental representation of the T -group:

$$\psi(x)' = T\psi(x) \quad (3.2)$$

$$\tilde{\psi}(x)' = \tilde{\psi}(x)T^t \quad (3.3)$$

Then by (2.1) the mass term is invariant. Here T is position independent.

The new idea that we are exploring is that the fields and the states lie in the T -algebra. Therefore suppose that ψ and $\tilde{\psi}$ have the following expansions

$$\psi = \psi^{(\beta)}\beta + \psi^{(\gamma)}\gamma \quad (3.4)$$

$$\tilde{\psi} = \beta\tilde{\psi}^{(\beta)} + \gamma\tilde{\psi}^{(\gamma)} \quad (3.5)$$

where $\psi^{(\beta)}$ and $\psi^{(\gamma)}$ as well as $\tilde{\psi}^{(\beta)}$ and $\tilde{\psi}^{(\gamma)}$ are two-dimensional vectors that do not lie in the algebra, and are orthogonal: $\tilde{\psi}^{(\beta)}\epsilon\psi^{(\gamma)} = \tilde{\psi}^{(\gamma)}\epsilon\psi^{(\beta)} = 0$. Note that $\tilde{\psi}$ is not ψ^t . Note that the action of T on ψ in (3.4) will carry ψ into other parts of the algebra (containing an infinite number of terms of the form $\beta^n\gamma^m\alpha^s\bar{\alpha}^\ell$). The general ψ therefore contains an infinite number of modes and Eq. (3.4) can be posited only in a special gauge. The expectation value of (3.1) in the state (nm) is

$$M[(\tilde{\psi}^{(\beta)}\epsilon\psi^{(\beta)})\langle nm|\beta^2|nm\rangle + (\tilde{\psi}^{(\gamma)}\epsilon\psi^{(\gamma)})\langle nm|\gamma^2|nm\rangle] . \quad (3.7)$$

Then the contribution of this term to the spectrum is, by (2.13) and (2.14)

$$M[f^{(\beta)}q^{2n}b^2 + f^{(\gamma)}q^{2m}c^2] \quad (3.8)$$

where

$$f^{(\beta)} = \tilde{\psi}^{(\beta)}\epsilon\psi^{(\beta)} \quad (3.9)$$

and there is a corresponding expression for $f^{(\gamma)}$. Set

$$f^{(\beta)}b^2 = \frac{1}{V} . \quad (3.10)$$

Then (3.8) may be written by (2.15) as follows:

$$M\left[q^{2n}\frac{1}{V} + q^{2m+2}(f^{(\beta)}f^{(\gamma)})V\right] . \quad (3.11)$$

The spectrum associated with M is inverted since $q < 1$. Eq. (3.11) bears some resemblance to the spectrum of the toroidally compactified string with its related large small (T) duality. By (2.11) however $n = m$. To obtain states $|nm\rangle$ with $n \neq m$ one may go to a higher rank quantum group.

The new states and levels may be likened to new states of polarization that depend on the tensor nature of ψ .

While q is dimensionless, there is no restriction on the dimensionality of the eigenvalues b and c . If b is a length, and V is a volume, then by (3.10) $\dim f^{(\beta)} = L^{-5}$ and $\dim f^{(\gamma)} = L^{-1}$.

For given n , the mass is minimized for the characteristic volume

$$\bar{V} = (q^2 f^{(\beta)} f^{(\gamma)})^{-1/2} . \quad (3.12)$$

Since $n = m$, (3.12) corresponds to the self-dual solution of (3.11).

In the limit of a simple field theory this spectrum collapses to a single level and the field quanta are point particles. In the present case the existence of any kind of mass spectrum may be interpreted to imply extension in configuration space. Assuming that the mass of a field quantum is given by (3.11) one may gain a qualitative idea of its spatial extension by noting that q^n differs little from a $\langle n \rangle_q$ spectrum—essentially the spectrum of a q -isotropic oscillator, with zero angular momentum. The wave functions of a q -oscillator are q -Hermite functions. The natural scale of the soliton would be fixed by (3.14).

If ψ is a Dirac spinor then the usual Lorentz invariant term is

$$M\psi^t C\psi \quad (3.15)$$

where C is the charge conjugation matrix that intertwines L^t and L^{-1}

$$L^t C L = L C L^t = C . \quad (3.16)$$

If it is supposed that the Lorentz symmetry is broken, one possibility is that L is replaced by the q -Lorentz group (L_q) which is defined by its spin representation as follows:

$$\epsilon \det_q L_q = L_q^t \epsilon L_q \quad \det_q L_q = 1 \quad (3.17)$$

Then (3.15) is replaced by

$$M\tilde{\psi}\epsilon\psi \quad (3.18)$$

and the new charge conjugation matrix is ϵ itself.

If this term is to be invariant under independent Lorentz and T transformation, we may write

$$M\tilde{\psi}C\epsilon\psi \quad (3.19)$$

where $\tilde{\psi}' = \tilde{\psi}L^t$ under Lorentz transformations and $\tilde{\psi}' = \tilde{\psi}T^t$ under T transformations.

4. Gauge Theory.

Much of the standard group theory, including orthogonality of irreducible representations, Clebsch-Gordan rules, etc., may be carried over to quantum groups. Closure, however, requires

$$(T_1 T_2)^t = T_2^t T_1^t \quad (4.1)$$

but (4.1) is guaranteed only if the matrix elements of T_1 commute with those of T_2 . To ensure this property one may take T_1 and T_2 at spatially (causally) separated points with respect to the light cone. The resulting groupoid is non-local.

When $T(x)$ depends on position, we may adopt the following Lagrangian which is both Lorentz and T -invariant

$$L = -\frac{1}{4} \sum_{\alpha} L(\alpha) \vec{F}_{\mu\nu} \vec{F}^{\mu\nu} R(\alpha) + i \tilde{\psi} C \epsilon \gamma^{\mu} \vec{\nabla}_{\mu} \psi + \frac{1}{2} [(\tilde{\varphi} \overleftarrow{\nabla}_{\mu}) \epsilon (\vec{\nabla}_{\mu} \varphi) + \tilde{\varphi} \epsilon \varphi] \quad (4.2)$$

where

$$\begin{aligned}
L(\alpha)' &= L(\alpha)T^{-1} & \varphi' &= T\varphi & (\vec{\nabla}\varphi)' &= T(\vec{\nabla}\varphi) \\
R(\alpha)' &= TR(\alpha) & \tilde{\varphi}' &= \tilde{\varphi}T^t & (\tilde{\varphi} \overleftarrow{\nabla})' &= (\varphi \overleftarrow{\nabla})T^t \\
(L\epsilon)' &= (L\epsilon)T^t & \psi' &= T\psi & (\vec{\nabla}\psi)' &= T(\vec{\nabla}\psi) \\
& & \tilde{\psi}' &= \tilde{\psi}T^t & (\tilde{\psi} \overleftarrow{\nabla})' &= (\tilde{\psi} \overleftarrow{\nabla})T^t
\end{aligned} \tag{4.3}$$

$$\begin{aligned}
\vec{\nabla}'_\mu &= T\vec{\nabla}_\mu T^{-1} , & \vec{\nabla}_\mu &= \vec{\partial}_\mu + \vec{A}_\mu , & \vec{F}_{\mu\nu} &= (\vec{\nabla}_\mu, \vec{\nabla}_\nu) \\
\overleftarrow{\nabla}'_\mu &= (T^t)^{-1} \overleftarrow{\nabla}_\mu T^t , & \overleftarrow{\nabla}_\mu &= \overleftarrow{\partial}_\mu + \overleftarrow{A}_\mu , & \overleftarrow{F}_{\mu\nu} &= (\overleftarrow{\nabla}_\mu, \overleftarrow{\nabla}_\nu)
\end{aligned}$$

Kinetic terms in $L(\alpha)$ and $R(\alpha)$, as well as other possible terms in \overleftarrow{A} have not been expressed in (4.2).

The invariance of the Lagrangian requires distinct left and right fields because F does not commute with T . In the limit $q = 1$ of (4.2) the L and R fields may be summed out as follows:

$$\sum_{\alpha} L_i(\alpha) R_j(\alpha) = \delta_{ij} \tag{4.4}$$

where the sum is over the complete set of left and right fields. Then

$$\lim_{q \rightarrow 1} \sum_{\alpha} L_i(\alpha) (FF)_{ij} R_j(\alpha) = \text{Tr } FF . \tag{4.5}$$

In this limit (4.2) becomes the usual Yang-Mills Lagrangian for $SU(2)$. In this limit also the internal structure described by the q -algebra of course disappears. The energy levels of the limiting Yang-Mills theory are therefore highly degenerate when viewed from within the $q = 1$ theory. This degeneracy is lifted when q is turned on.

5. Representation of the Free Fields.

After expanding the space of one-particle states we may adopt the conventional representations of the free scalar and spinor fields:

scalar

$$\varphi(x) = \left(\frac{1}{2\pi}\right)^{3/2} \int \frac{d\vec{p}}{(2p_o)^{1/2}} \sum_s u(p, s) [e^{-ipx} a(p, s) + e^{ipx} \bar{a}(p, s)] \tau_s \tag{5.1}$$

spinor

$$\psi(x) = \left(\frac{1}{2\pi}\right)^{3/2} \int \frac{d\vec{p}}{(2p_o)^{1/2}} \sum_{r,s} [u(p, r, s) e^{-ipx} a(p, r, s) + v(p, r, s) e^{ipx} \bar{b}(p, r, s)] \tau_s \tag{5.2}$$

where the only new element added to the conventional expansions is the sum over τ_s where τ_s lies in the q -algebra.

The corresponding choice for the vector fields offers more possibilities even if we are guided by the standard non-Abelian theory employing the following representation in the $SU(2)$ theory for the vector W_μ :

$$W_\mu = W_\mu(+)\tau(-) + W_\mu(-)\tau(+) + W_\mu(3)\tau_3 . \quad (5.3)$$

In the $SU_q(2)$ theory one option is based on the following correspondence between the q -algebra and the Cartan subalgebra of $SU(2)$:

$$\bar{\alpha} \sim E_+ \quad \alpha \sim E_- \quad \text{and} \quad \beta, \gamma \sim H \quad (5.4)$$

Then one may propose

$$W_\mu^{(q)} = W_\mu^+ \alpha + W_\mu^- \bar{\alpha} + W_\mu^{(\beta)} \beta + W_\mu^{(\gamma)} \gamma \quad (5.5)$$

where the coefficients are conventional. However, one may also deform the Lie algebra of $SU(2)$. Then the generators of the q -Lie algebra satisfy

$$\begin{aligned} (J_3^{(q)}, J_\pm^{(q)}) &= \pm J_\pm^q \\ (J_+^{(q)}, J_-^{(q)}) &= \frac{1}{2}[2J_3^{(q)}]_q \end{aligned} \quad (5.6)$$

where

$$[2J_3^{(q)}]_q = \frac{q^{2J_3^q} - q^{-2J_3^q}}{q - q^{-1}}$$

and instead of (5.5) one may choose

$$W_\mu^{(q)} = W_\mu^{(+)} J(-)^q + W_\mu^{(-)} J(+)^q + W_\mu(3) J_3^q . \quad (5.7)$$

In constructing the q -theory one may choose either option. It is more natural, however, to take the view that the two options are not mutually exclusive and that the q -theory implies a deformation of both the group (g) and the algebra (u).

The q -theory describing deformations of both g and u would therefore be composed of two sectors. The first sector would contain the new particles lying in the algebra of $(\alpha, \bar{\alpha}, \beta, \gamma)$. The second sector would lie close to the standard theory but would predict differences from the standard theory that depend on q and these predictions would be accessible by perturbation theory since $q < 1$. In the $q = 1$ correspondence limit one would recover the standard theory while the new particles lying in the first sector would disappear as q approaches unity and in any case would be more difficult to detect.

I thank C. Fronsdal and V. S. Varadarajan for useful comments.

References.

1. N. Yu. Reshetikhin, L. A. Takhtadzhyan, and L. D. Fadeev, Leningrad Math. J. **1** (1990) No. 1.
2. The literature on quantum groups is extensive. See for example, *Quantum Groups*, edited by T. Curtwright, D. Fairlee and C. Zachos, World Scientific (1991).
3. R. Finkelstein, Observable Properties of q -Deformed Physical Systems, Lett. Math. Phys. **49**, 105 (1999); Gauged q -Fields, hep-th/9906135; Spontaneous Symmetry Breaking of q -Gauge Field Theory, hep-th/9908210.